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Explosion for Some Semilinear Wave Equations

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In this note we show that known results on the propagation of singularities for semilinear wave equations suffice for the construction of explosive solutions with arbitrarily small data. In particular, for the quadratically nonlinear wave equation in one space dimension

$$u_{tt} - u_{xx} = Au_t^2 + Bu_t u_x + Cu_x^2 \equiv Q(u_t, u_x) \tag{1}$$

we show that the only nonlinearity without such explosions is Nirenberg's equation $B = 0$, $C = -A$ which is transformed to the linear wave equation by the substitution $v = e^{Au}$. This nonlinearity is also singled out by special compactness properties [3] and its role in the equipartition of energy [1]. Another much studied equation for which we supply explosive solutions is the Carleman model of transport theory, $\partial_\pm \equiv \partial_t \pm \partial_x$,

$$\partial_\pm u_\pm = \mp(u_+^2 - u_-^2). \tag{2}$$

Here there is global solvability forward in time for positive data. We have small positive data which explode in the past and small nonpositive data which explode in the future. Our method, in a word, is to construct solutions with jump discontinuities (jumps in $\nabla_{t,x} u$ for (1)) and show that the jump explodes.

Both (1) and (2) are special cases. The first is converted to a system for $u_\pm \equiv \partial_\mp u$,

$$\partial_\pm u_\pm = Q((u_+ + u_-)/2, (u_- - u_+)/2).$$

The general quadratically nonlinear 2×2 system containing these two examples is

$$\partial_\pm u_\pm = f_\pm(u_+, u_-) \tag{3}$$

$$f_\pm = A_\pm u_\pm^2 + B_\pm u_+ u_- + C_\pm u_+^2. \tag{4}$$

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Even for general functions $f_\pm$ the initial valve problem for (3) is well set, locally, for data in $L^\infty(\mathbb{R})$. That is given $u^0_\pm \in L^\infty(\mathbb{R})$ there is an interval $(a, b) \ni 0$ and a unique $u \in L^\infty_{\text{loc}}((a, b) : L^\infty(\mathbb{R}))$ solution to (3) with

$$u_\pm(0, \cdot) = u^0_\pm(\cdot).$$

In addition, the map $t \to u_\pm(t, \cdot)$ is continuous with values in $L^\infty_*(\mathbb{R})$, that is L^∞ endowed with the weak star topology. The interval of existence can be chosen uniform for data in bounded subsets of L^∞ so the maximal interval of existence (a^*, b^*) is characterized by

$$b^* = \infty \qquad \text{or} \qquad \lim_{t \uparrow b^*} \|u(t, \cdot)\|_{L^\infty} = \infty$$

with a similar assertion as $t \searrow a^*$. For data of the form $\varepsilon u^0_\pm$ one finds existence in an interval with $\min(|a^*|, b^*) > \text{const}/\varepsilon$.

THEOREM 1. *Suppose that in* (3, 4) $A_+ \neq 0$ *or* $C_- \neq 0$. *Then there exist* $u^0_\pm \in L^\infty(\mathbb{R})$, *piecewise* C^∞ *and of compact support such that the initial value problem for* (3), (4) *with data* $u^0_\pm(0, \cdot) = \varepsilon u^0_\pm(\cdot)$ *does not have a global solution for any* $\varepsilon \neq 0$. *In fact, the maximal interval of existence* $(a^*_\varepsilon, b^*_\varepsilon)$ *satisfies* $\min(|a^*_\varepsilon|, b^*_\varepsilon) \leqslant \text{const}/\varepsilon$.

Remark 1. It is natural to compare this result to [2, Theorem 4]. Our condition is weaker than that of the second half of Theorem 4 but we do not construct C^∞ solutions which explode.

Remark 2. If $A_\pm = C_\pm = 0$, L. Tartar [7], has proved global solvability for data small in $L^\infty(\mathbb{R})$. Combining our result with that of [2], the undecided cases are when the following three conditions hold simultaneously: $A_+ = C_- = 0$, $B^2_+ + B^2_- \neq 0$, $f_+ \not\equiv 0$ and $f_- \not\equiv 0$.

We construct explosive solutions, without any additional effort for systems of the form (3) with

$$f_+(u_+, 0) \equiv g_+(u_+), \qquad f_-(0, u_-) \equiv g_-(u_-), \tag{5}$$

where $f_\pm$ are C^∞ on $\mathbb{R} \times \mathbb{R}$, $g_\pm(0) = 0$, and one of $g_\pm$ satisfy

(i) $sg(s) > 0$ for $s \neq 0$,

(ii) $\int^\infty 1/g(s) < \infty$ and $\int_\infty 1/|g(s)|\, ds < \infty$.

In this case the time of explosion is comparable to the time of explosion of the ordinary differential equation $\psi' = g(\psi)/\sqrt{2}$, $\psi(0) = c\varepsilon$.

THEOREM 2. *If the initial data* $u^0 = (u^0_+, u^0_-) \in L^\infty(\mathbb{R})^2$ *is piecewise* *smooth with support in* $x \leqslant \bar{x}$ *then the* $L^{\infty}_{\text{loc}}((a^*, b^*): L^{\infty}(\mathbb{R}))$ *solution,* u, *of* (3) *with* $u(0, \cdot) = u^0$ *satisfies*

$$\|u_{\pm}(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \geqslant \psi_{\pm}(t) \qquad \textit{for} \quad \pm t > 0,$$

where $\psi_{\pm}$ *are the solutions to*

$$\psi'_{\pm} = \pm g_{\pm}(\psi)/\sqrt{2}, \qquad \psi_{\pm}(0) = u^0_{\pm}(\bar{x}-).$$

Remark 1. The value $u^0_{\pm}(\bar{x}-) = \lim_{\varepsilon \searrow 0} u^0_{\pm}(\bar{x} - \varepsilon)$ is the limit of $u^0_{\pm}$ from the left at $\bar{x}$. Since $u^0_{\pm}$ vanishes for $x > \bar{x}$ it also the value of the jump of $u^0_{\pm}$ at $\bar{x}$.

Remark 2. There is, of course, an analogue for data supported $x \geqslant \bar{x}$.

Proof of Theorem 1 *assuming Theorem* 2. One need only take $u^0(\cdot)$ as in Theorem 2 with one of the quantities $A_+ u^0_+(\bar{x}-)$, $-C_- u^0_-(\bar{x}-)$ strictly positive. The corresponding $\psi_{\pm}$ explodes in time $\pm t = \text{const}/\varepsilon$. ▮

Proof of Theorem 2. Theorem 1 of [5] shows that u is piecewise smooth in $(a^*, b^*) \times \mathbb{R}$ with singularities only along the characteristics issuing from singularities of u^0. Here it is crucial that we have no more than two sound speeds.

We study the jump in u along $\Gamma_{\pm} = \{(t, x): x = \bar{x} \pm t, \pm t \geqslant 0\}$. Finite propogation speed implies that $u \equiv 0$ to the right of $\Gamma_{\pm}$.

We consider first Γ_+. Since $\partial_- u_-$ is locally bounded, we see that u_- does not jump accorss Γ_+. Thus the jump at $p \in \Gamma_+$ is given by

$$j(p) = \lim_{\varepsilon \searrow 0} u_+(p - \varepsilon(0, 1)).$$

The transport equation for the jump j along Γ_+ is then

$$(\partial_t + \partial_x) j = [f_+(u_+, u_-)]$$

when [ ] signifies jump, left hand limit minus right hand limit. As u_- does not jump and vanishes on the right, it also vanishes on the left so the right side is equal to $[g_+(u_+)] = g_+(j)$, the last equality since $u_+ = 0$ to the right and $g(0) = 0$. Thus for $t > 0$ we have $j(t, \bar{x} + t) = \psi_+(t)$ and

$$\|u_+(t)\|_{L^{\infty}} \geqslant |j(t, \bar{x} + t)| = \psi_+(t)$$

finishes the proof for $t > 0$.

For $t < 0$, the jump j in u across Γ_- satisfies $(\partial_t - \partial_x) j = g_-(j)$ and the estimate for $t < 0$ follows. ▮

We next discuss the implications of these results in dimensions greater than one, restricting attention to the semilinear wave equation

$$\Box u = Q(u_t, \nabla_x u), \qquad Q \neq c(u_t^2 - |\nabla_x u|^2). \tag{6}$$

Looking for solutions which depend only on t, $\omega \cdot x$, $|\omega| = 1$ leads one to the one dimensional equation, and choosing ω appropriately the one dimensional equation will have nonlinearity not equal to $c(u_t^2 - u_x^2)$. Rotating coordinates we may suppose that $\omega = (1, 0, ..., 0)$. One then can find $\bar{u}^\varepsilon(t, x) = \bar{u}^\varepsilon(t, x_1)$ with initial data

$$\bar{u}^\varepsilon(0, x) = \varepsilon\varphi(x_1), \qquad \bar{u}_t^\varepsilon(0, x) = \varepsilon\psi(x_1),$$

with φ, ψ, $\nabla\varphi$ supported in $x_1 \leqslant \bar{x}_1$, piecewise smooth, singular only at $x_1 = \bar{x}$, and with $\nabla_{t,x}\bar{u}^\varepsilon$ exploding along $x_1 = \bar{x}_1 + t$, $t > 0$ in time $\leqslant$ const/ε.

We next modify the solutions to achieve the same explosion for data of compact support. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ in the neighborhood of $(\bar{x}_1, 0, 0, ..., 0)$. Let u^ε be the solution of (6) with initial data

$$u^\varepsilon(0, x) = \varepsilon\chi(x)\,\varphi(x_1), \qquad u_t^\varepsilon(0, x) = \varepsilon\chi(x)\,\psi(x_1).$$

For such piecewise smooth data it is proved in [6, Theorem 7, 8] that there is a piecewise smooth solution u^ε on $(a_\varepsilon^*, b_\varepsilon^*) \times \mathbb{R}^n$ with singularities only on $\{x_1 = \bar{x}_1 \pm t\}$. The solution is unique in the class of functions with $\nabla_{t,x}u \in L^\infty((a_\varepsilon^*, b_\varepsilon^*) \times \mathbb{R}^n)$ and the interval of existence $(a_\varepsilon^*, b_\varepsilon^*) \ni 0$ is characterized by

$$b_\varepsilon^* = \infty \qquad \text{or} \qquad \lim_{t \uparrow b_\varepsilon^*} \|\nabla_{t,x}u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} = \infty$$

with a similar assertion as $t \searrow a_\varepsilon^*$. In addition, it is not hard to show that

$$\min(|a_\varepsilon^*|, b_\varepsilon) \geqslant \text{const}/\varepsilon.$$

On the other hand, a simple domain of dependence argument shows that $u^\varepsilon = \bar{u}^\varepsilon$ in the neighborhood of the ray $(t, \bar{x}_1 + t, 0, ..., 0)$ and therefore $\|\nabla_{t,x}u^\varepsilon\|_{L^\infty(\mathbb{R}^n)}$ explodes in time $b^* \leqslant$ const/ε.

In interpreting the exploding singularities above one must bear in mind that the construction is made in the piecewise smooth category with singularities along regular hypersurfaces. In that category $\|\nabla_{t,x}u\|_{L^\infty(\mathbb{R}^n)}$ is natural. Recall that for $n > 1$ there are solutions of $\Box w = 0$ with $\nabla_{t,x}w(0, \cdot) \in L^\infty$ and $\|\nabla_{t,x}w(t, \cdot)\|_{L^\infty} = \infty$ for all $t \neq 0$. On the other hand, for piecewise smooth data ψ, $\nabla\varphi$ singular on $\{x_1 = \bar{x}_1\}$, the L^∞ norm of the gradient does not explode in the linear case and, therefore, does not explode for Nirenberg's nonlinearity and small data. Thus, the constructions above are of some interest.

The last example contrasts sharply with the results of Klainerman [4] and their later refinements which guarantee existence of global solutions for small data provided the number of space dimensions is at least four. Those results require data, φ, ψ which are quite regular. Roughly speaking, the singular solutions above avoid the dispersion which is the key to Klainerman's results. In fact, the rays $(t, \bar{x}_1 + t, x_2, ..., x_n)$ with $|x_2, ..., x_n|$ small, which are the carriers of the singularities, do not disperse at all. If one tries to make the same constructions with singularities on an outgoing light cone, the rays disperse and the singularities can explode only for $n \leqslant 3$.

One can even consider the radial equation

$$u_{tt} - \frac{1}{r^{n-1}} \partial_r r^{n-1} \partial_r u = Q(u_t, u_r) \neq c(u_t^2 - u_r^2)$$

for fractional dimension n and one finds exploding singularities exactly for $n \leqslant 2\sqrt{2} + 1$. This somewhat mysterious result follows from the fact that the jumps satisfy transport equations of the form

$$\sqrt{2}\, \psi' \pm \frac{n-1}{2t} \psi + c\psi^2 = 0$$

which have exploding solutions for small data if and only if $n \leqslant 2\sqrt{2} + 1$. The details of these last computations are omitted.

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